Separation Axioms and Connectedness for βω−Open Sets

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ABSTRACT

Our propose in this paper is to introduce the new classes for separation axioms in topological spaces by using $\beta\omega-$open sets and $G_{\beta\omega}-$open sets, called $\beta\omega-$separation axioms and $G_{\beta\omega}-$separation axioms. Furthermore, we introduce the stronger form of connected spaces.

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Keywords

Open set; Generalized closed set; Connectedness.

1. INTRODUCTION

In 1982 Hdeib [4] introduced the notion of a $\omega-$open sets. A subset A of a space X is called $\omega-$open set if for each $x \in A$, there is an open set U_x containing x such that $U_x - A$ is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set. In 1983 the authors [7] introduced the weak form for an open set which is called a $\beta-$open set. A subset A of a space X is called a $\beta-$open set if $A \subseteq Cl(Int(Cl(A)))$. The complement of a $\beta-$open set is called a $\beta-$closed set. In 2005 Al-Zoubi [6] introduced the generalization property of $\omega-$open sets. A subset A of a space X is called generalized $\omega-$closed set if $Cl_{\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open set. The complement of generalized $\omega-$closed set is called generalized $\omega-$open set, where $Cl_{\omega}(A)$ is the $\omega-$closure set of A. In 2009 Noiri and Noorani [9] introduced the notion of $\beta\omega-$open set as weak form for a $\omega-$open sets and a $\beta-$open sets. A subset A of a space X is called a $\beta\omega-$open set if $A \subseteq Cl(Int_{\omega}(Cl(A)))$. The complement of a $\beta\omega-$open set is called a $\beta\omega-$closed set, where $Int_{\omega}(A)$ is the $\omega-$interior set of A. In 2020 [1] we introduced the notion of $G_{\beta\omega}-$closed set as weak form for a $\beta\omega-$closed sets and a $\beta-$open sets. A subset A of a topological space (X, τ) is called generalized $\beta\omega-$closed (simply $G_{\beta\omega}-$closed) set if $Cl_{\beta\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ). The complement of $G_{\beta\omega}-$closed set is called generalized $\beta\omega-$open (simply $G_{\beta\omega}-$open) set, where $Cl_{\beta\omega}(A)$ is the $\beta\omega-$closure set of A which defined as the intersection of all $\beta\omega-$closed subsets of X containing A. Similar, the $\beta\omega-$interior set of A is defined as the union of all $\beta\omega-$open subsets of X contained in A and is denoted by $Int_{\beta\omega}(A)$.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we introduce the new classes for separation axioms in topological spaces, called $\beta\omega-$separation axioms. Furthermore, the relationship with the other known axioms will be studied. In Section 4 we introduce also the new classes for separation axioms in topological spaces, called $G_{\beta\omega}-$separation axioms. Furthermore, the relationship with the other known axioms will be also studied. In Section 5 we introduce the stronger form of connected spaces.

2. PRELIMINARIES

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

A subset of topological space is called a *clopen* set if it is both open and closed set. A topological space (X, τ) is called *0-dimensional space* [3] if it has a base consisting clopen sets. A topological space (X, τ) is called a *disconnected space* [3] if it is the union of two nonempty subsets A and B such that $Cl(A) \cap B = \emptyset$ and $A \cap Cl(B) = \emptyset$.

THEOREM 2.1. [3] A topological space (X, τ) is a disconnected space if and only if it is the union of two disjoint nonempty open subsets.

THEOREM 2.2. [3] For a topological space (X, τ) and $A, B \subseteq X$, if B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

THEOREM 2.3. [8] Every closed set is a $g-$closed set.

DEFINITION 2.4. [8] A topological space (X, τ) is called a $T_{1/2}-space$ if every $g-$closed set is closed set.

THEOREM 2.5. [5] A topological space (X, τ) is $T_{1/2}-space$ if and only if every singleton set is open or closed set.

THEOREM 2.6. [1] Let (X, τ) be a topological space. If (X, τ) is a $T_{1/2}-$space then every $G_{\beta\omega}-$closed set in X is $\beta\omega-$closed set in X.

THEOREM 2.7. [3] A topological space (X, τ) is T_1-space if and only if every singleton set is closed set.

THEOREM 2.8. [3] A topological space (X, τ) is regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is an open set M in X containing x such that $Cl(M) \subseteq N$.

DEFINITION 2.9. [1] A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a space (X, τ) into a space (Y, ρ) is called:

(1) $\beta\omega-continuous\ function$ if $f^{-1}(U)$ is a $\beta\omega-$open set in X for every open set U in Y.

(2) $G_{\beta\omega}-continuous\ function$ if $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X for every open set U in Y.

3. $\beta\omega-$SEPARATION AXIOMS

DEFINITION 3.1. A topological space (X, τ) is called:

(1) $\beta_\omega^2-space$ if for two points $x \neq y \in X$ in X, there are two $\beta\omega-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.

(2) $\beta\omega-regular\ space$ if for each closet set F in X and each $x \notin F$, there are two $\beta\omega-$open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $\beta_\omega^3-space$ if it is $\beta\omega-$regular space and T_1-space.

(3) $\beta\omega-normal\ space$ if for each two disjoint closet sets F and M in X, there are two $\beta\omega-$open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $\beta_\omega^4-space$ if it is $\beta\omega-$normal space and T_1-space.

The proof of the following theorem, Theorem (3.3) and Theorem (3.4) follow from the fact that open sets are $\beta\omega-$open sets.

THEOREM 3.2. Every T_2-space is a β_ω^2-space.

THEOREM 3.3. Every regular space is a $\beta\omega-$regular space.

THEOREM 3.4. Every normal space is a $\beta\omega-$normal space.

The converse of the Theorems (3.2), (3.3) and (3.4) are no need to be true.

EXAMPLE 3.5. Let $X = \{1, 2, 3\}$. The indiscrete topological space (X, T_I), where $T_I = \{\emptyset, X\}$, is β_ω^2-space, $\beta\omega-$regular space and $\beta\omega-$normal space, since all subsets of countable topological space are $\beta\omega-$open sets, but it is not T_2-space, regular space or normal space.

THEOREM 3.6. Every β_ω^3-space is a β_ω^2-space.

PROOF. Let (X, τ) be a β_ω^3-space and $x \neq y \in X$ be any points in X. Since X is a T_1-space then by Theorem (2.7), $\{x\}$ is a closed set in X and $y \notin \{x\}$. Since X is a $\beta\omega-$regular space then there are two $\beta\omega-$open sets G and U in X such that $x \in \{x\} \subseteq G$, $y \in U$ and $U \cap G = \emptyset$. Hence X is a β_ω^2-space. $\square$

THEOREM 3.7. Every β_ω^4-space is a β_ω^3-space.

PROOF. Let (X, τ) be a β_ω^4-space. Let F be any closed set in X and $x \notin F$ be any points in X. Since X is a T_1-space then by Theorem (2.7), $\{x\}$ is a closed set in X and $F \cap \{x\} = \emptyset$. Since X is a $\beta\omega-$normal space then there are two $\beta\omega-$open sets G and U in X such that $x \in \{x\} \subseteq G$, $F \subseteq U$ and $U \cap G = \emptyset$. Hence X is a β_ω^3-space. $\square$

THEOREM 3.8. A topological space (X, τ) is a β_ω^2-space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $\beta\omega-$open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.

PROOF. Suppose that (X, τ) is β_ω^2-space. Let $x \in X$ be any point in X and $y \neq x$ be any points in X. Then there are two $\beta\omega-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Take $M = G$ is a $\beta\omega-$open set in X containing x and so $y \notin M \subseteq Cl_{\beta\omega}(M)$.

Conversely, Let $x \neq y \in X$ be any points in X. By the hypothesis, there is a $\beta\omega-$open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$. Then $X - Cl_{\beta\omega}(M)$ is a $\beta\omega-$open set M in X containing y and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$. Then (X, τ) is β_ω^2-space. $\square$

THEOREM 3.9. A topological space (X, τ) is a $\beta\omega-$regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is a $\beta\omega-$open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.

PROOF. Suppose that (X, τ) is $\beta\omega-$regular space. Let $x \in X$ be any point in X and N be any open set in X containing x. Then $X - N$ is a closed set in X and $x \notin X - N$. Since (X, τ) is $\beta\omega-$regular space then there are two $\beta\omega-$open sets G and U in X such that $X - N \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Take $M = U$ is a $\beta\omega-$open set in X containing x. Then $M = U \subseteq X - G$, this implies, $Cl_{\beta\omega}(M) \subseteq Cl_{\beta\omega}(X - G) = X - G \subseteq N$.
Conversely, Let F be any closed set in X and $x \notin F$. Then $x \in X - F$ and $X - F$ is an open set in X containing x. By the hypothesis, there is a $\beta\omega-$open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(M)$ and $X - Cl_{\beta\omega}(M)$ is a $\beta\omega-$open set in X. Since M is a $\beta\omega-$open set in X containing x and $M \cap [X - Cl_{\beta\omega}(M)] = \emptyset$, then (X, τ) is $\beta\omega-$regular space. $\square$

THEOREM 3.10. A topological space (X, τ) is $\beta\omega-$normal space if and only if for each closed set F in X and for each open set G in X containing F, there is a $\beta\omega-$open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.

PROOF. Suppose that (X, τ) is $\beta\omega-$normal space. Let F be any closed set in X and G be any open set in X containing F. Then $X - G$ is a closed set in X and $F \cap (X - G) = \emptyset$. Since (X, τ) is $\beta\omega-$normal space then there are two $\beta\omega-$open sets H and U in X such that $X - G \subseteq U$, $F \subseteq H$ and $U \cap H = \emptyset$. Take $V = H$ is a $\beta\omega-$open set in X containing F. Then $V = H \subseteq X - U$, this implies, $Cl_{\beta\omega}(V) \subseteq Cl_{\beta\omega}(X - U) = X - U \subseteq G$.
Conversely, Let F and M be any two closed sets in X such that $F \cap M = \emptyset$. Then $M \subseteq X - F$ and $X - F$ is an open set in X containing closed set M. By the hypothesis, there is a $\beta\omega-$open set V in X containing M such that $Cl_{\beta\omega}(V) \subseteq X - F$. Then $F \subseteq X - Cl_{\beta\omega}(V)$ and $X - Cl_{\beta\omega}(V)$ is a $\beta\omega-$open set in X. Since V is a $\beta\omega-$open set in X containing x and $V \cap [X - Cl_{\beta\omega}(V)] = \emptyset$, then (X, τ) is $\beta\omega-$normal space. $\square$

THEOREM 3.11. If a function $f : (X, \tau) \rightarrow (Y, \rho)$ is an injection $\beta\omega-$continuous and Y is a T_2-space then X is a β_ω^2-space.

PROOF. Let Y be a T_2-space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $\beta\omega-$continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega-$open sets in X. Hence X is a β_ω^2-space. $\square$

A subset of topological space is called a $\beta\omega-clopen$ set if it is both $\beta\omega-$open and $\beta\omega-$closed set. sets.

DEFINITION 3.12. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is called *slightly $\beta\omega-continuous\ function$* if $f^{-1}(U)$ is a $\beta\omega-$clopen set in X for every clopen set U in Y.

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THEOREM 3.13. Let $f : (X, \tau) \to (Y, \rho)$ be an injection slightly $\beta\omega-$continuous function and Y be 0-dimensional. If Y is a T_2-space then X is a β_ω^2-space.

PROOF. Let Y be a T_2-space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Since Y is 0-dimensional space there are two clopen sets G_1 and U_1 in Y such that $f(x) \subseteq G_1 \subseteq G$ and $f(y) \subseteq U_1 \subseteq U$. Then $x \subseteq f^{-1}(G_1) \subseteq f^{-1}(G)$ and $y \subseteq f^{-1}(U_1) \subseteq f^{-1}(U)$. Then and

$$f^{-1}(G_1) \cap f^{-1}(U_1) \subseteq f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = \emptyset.$$

Since G_1 and U_1 are clopen sets in Y and f is a slightly $\beta\omega-$continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $\beta\omega-$open sets in X. Hence X is a β_ω^2-space. $\square$

THEOREM 3.14. Let $f : (X, \tau) \to (Y, \rho)$ be an injection $\beta\omega-$continuous function. If f is an open (or closed) function and Y is a regular space then X is a $\beta\omega-$regular space.

PROOF. Firstly suppose f is an open function. Let $x \in X$ be any point in X and U be any open set containing x. Then $f(x) \in f(U)$ and $f(U)$ is an open set in Y. Since Y is a regular space then by Theorem(2.8), there is an open set M in Y containing $f(x)$ such that $Cl(M) \subseteq f(U)$. Since f is a $\beta\omega-$continuous then $V = f^{-1}(M)$ is a $\beta\omega-$open set in X containing x. Since f is injection then $f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U$. Then

$$Cl_{\beta\omega}(V) = Cl_{\beta\omega}[f^{-1}(M)] \subseteq f^{-1}[Cl(M)] \subseteq f^{-1}[f(U)] \subseteq U.$$

Hence by Theorem (3.9), X is a $\beta\omega-$regular space. Secondly, similar, if f is a closed function. $\square$

THEOREM 3.15. Let $f : (X, \tau) \to (Y, \rho)$ be an injection slightly $\beta\omega-$continuous and Y is 0-dimensional space. If f is an open (or closed) function then X is a $\beta\omega-$regular space.

PROOF. Firstly suppose f is a closed function. Let F be any closed set in X and $x \notin F$. Then $f(x) \notin f(F)$ and $f(F)$ is a closed set in Y. Then $f(x) \in Y - f(F)$ and $Y - f(F)$ is an open set in Y. Since Y is a 0-dimensional space then there is a clopen set V in Y such that $f(x) \in V \subseteq Y - f(F)$. Since f is injection then

$$x \in f^{-1}(V) \subseteq f^{-1}[Y - f(F)] \subseteq X - F.$$

Since f is a slightly $\beta\omega-$continuous then $f^{-1}(V)$ is a $\beta\omega-$clopen set in X containing x and $X - f^{-1}(V)$ is a $\beta\omega-$clopen set in X such that $F \subseteq X - f^{-1}(V)$ Hence X is a $\beta\omega-$regular space. Secondly, similar, if f is open function. $\square$

THEOREM 3.16. Let $f : (X, \tau) \to (Y, \rho)$ be an injection $\beta\omega-$continuous function. If f is closed function and Y is a normal space then X is a $\beta\omega-$normal space.

PROOF. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. since Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $\beta\omega-$open in X. Hence X is a $\beta\omega-$normal space. $\square$

THEOREM 3.17. Let $f : (X, \tau) \to (Y, \rho)$ be an injection slightly $\beta\omega-$continuous and Y is 0-dimensional space. If f is a closed function and Y is a normal space then X is a $\beta\omega-$normal space.

PROOF. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since Y is a 0-dimensional space then for every $g \in f(F)$ and $u \in f(H)$ there are clopen sets U_u and G_g in Y such that $u \in U_u \subseteq U$ and $g \in G_g \subseteq G$. Then

$$f(H) \subseteq \cup \{U_u : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\} \subseteq U$$

and

$$f(F) \subseteq \cup \{G_g : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\} \subseteq G.$$

This implies,

$$H \subseteq \cup \{f^{-1}(U_u) : u \in f(H) \text{ and } U_u \text{ is a clopen set in } Y\}$$

$$\subseteq f^{-1}(U) \text{ and}$$

$$F \subseteq \cup \{f^{-1}(G_g) : g \in f(F) \text{ and } G_g \text{ is a clopen set in } Y\}$$

$\subseteq f^{-1}(G)$. Since f is a slightly $\beta\omega-$continuous then $f^{-1}(U_u)$ and $f^{-1}(G_g)$ are $\beta\omega-$open in X for all $g \in f(F)$ and $u \in f(H)$. So that $M = \cup \{f^{-1}(U_u) : u \in f(H)\}$ and $N = \cup \{f^{-1}(G_g) : g \in f(F)\}$ are $\beta\omega-$open in X and

$$M \cap N \subseteq f^{-1}(U) \cap f^{-1}(G) \subseteq f^{-1}(U \cap G) = f^{-1}(\emptyset) = \emptyset.$$

Hence X is a $\beta\omega-$normal space. $\square$

4. $G_{\beta\omega}-$SEPARATION AXIOMS

DEFINITION 4.1. A topological space (X, τ) is called:

(1) $G_{\beta\omega}^2-$*space* if for two points $x \neq y \in X$ in X, there are two $G_{\beta\omega}-$open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$.

(2) $G_{\beta\omega}-$*regular space* if for each closet set F in X and each $x \notin F$, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $G_{\beta\omega}^3-$*space* if it is $G_{\beta\omega}-$regular space and T_1-space.

(3) $G_{\beta\omega}-$*normal space* if for each two disjoint closet sets F and M in X, there are two $G_{\beta\omega}-$open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. A topological space (X, τ) is called $G_{\beta\omega}^4-$*space* if it is $G_{\beta\omega}-$normal space and T_1-space.

It is clear that every β_ω^2-space is a $G_{\beta\omega}^2-$space, every $\beta\omega-$regular space is a $G_{\beta\omega}-$regular space and every $\beta\omega-$normal space is a $G_{\beta\omega}-$normal space.

THEOREM 4.2. Every $G_{\beta\omega}^3-$space is a $G_{\beta\omega}^2-$space.

PROOF. Similar for Theorem (3.6). $\square$

THEOREM 4.3. Every $G_{\beta\omega}^4-$space is a $G_{\beta\omega}^3-$space.

PROOF. Similar for Theorem (3.7). $\square$

THEOREM 4.4. Let (X, τ) be a $T_{1/2}-$space. If X is a $G_{\beta\omega}^2-$space then X is a β_ω^2-space.

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PROOF. For two points $x \neq y \in X$ in X, since X is a $G_{\beta\omega}^2$−space, there are two $G_{\beta\omega}$−open sets G and U in X such that $x \in G$, $y \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$−space, then by Theorem (2.6), G and U are $\beta\omega$−open sets in X. Hence X is a β_ω^2−space. □

THEOREM 4.5. Let (X, τ) be a $T_{1/2}$−space. If X is a $G_{\beta\omega}$−regular space then X is a $\beta\omega$−regular space

PROOF. For each closet set F in X and each $x \notin F$, since X is a $G_{\beta\omega}$−regular space, there are two $G_{\beta\omega}$−open sets G and U in X such that $F \subseteq G$, $x \in U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$−space, then by Theorem (2.6), G and U are $\beta\omega$−open sets in X. Hence X is a $\beta\omega$−regular space. □

COROLLARY 4.6. Every $G_{\beta\omega}^3$−space is a β_ω^3−space.

PROOF. Use above theorem, since every T_1−space is $T_{1/2}$−space. □

THEOREM 4.7. Let (X, τ) be a $T_{1/2}$−space. If X is a $G_{\beta\omega}$−normal space then X is a $\beta\omega$−normal space

PROOF. For each two disjoint closet sets F and M in X, since X is a $G_{\beta\omega}$−normal space, there are two $G_{\beta\omega}$−open sets G and U in X such that $F \subseteq G$, $M \subseteq U$ and $U \cap G = \emptyset$. Since X is a $T_{1/2}$−space then by Theorem (2.6), G and U are $\beta\omega$−open sets in X. Hence X is a $\beta\omega$−normal space. □

COROLLARY 4.8. Every $G_{\beta\omega}^4$−space is a β_ω^4−space.

PROOF. Use above theorem, since every T_1−space is $T_{1/2}$−space. □

THEOREM 4.9. A topological space (X, τ) is $G_{\beta\omega}^2$−space if and only if for each $x \in X$ and for $y \neq x \in X$, there is a $G_{\beta\omega}$−open set M in X containing x such that $y \notin Cl_{\beta\omega}(M)$.

PROOF. Similar for Theorem (3.8). □

THEOREM 4.10. A topological space (X, τ) is $G_{\beta\omega}$−regular space if and only if for each $x \in X$ and for each open set N in X containing x, there is a $G_{\beta\omega}$−open set M in X containing x such that $Cl_{\beta\omega}(M) \subseteq N$.

PROOF. Similar for Theorem (3.9). □

THEOREM 4.11. A topological space (X, τ) is $G_{\beta\omega}$−normal space if and only if for each closed set F in X and for each open set G in X containing F, there is a $G_{\beta\omega}$−open set V in X containing F such that $Cl_{\beta\omega}(V) \subseteq G$.

PROOF. Similar for Theorem (3.10). □

THEOREM 4.12. If a function $f : (X, \tau) \to (Y, \rho)$ is $G_{\beta\omega}$−continuous injection and Y is a T_2−space then X is a $G_{\beta\omega}^2$−space.

PROOF. Let Y be a T_2−space and $x \neq y \in X$ be any points in X. Since f is injection then $f(x) \neq f(y) \in Y$. Then there are two open sets G and U in Y such that $f(x) \in G$, $f(y) \in U$ and $U \cap G = \emptyset$. Then $x \in f^{-1}(G)$, $y \in f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since G and U are open sets in Y and f is a $G_{\beta\omega}$−continuous then $f^{-1}(U)$ and $f^{-1}(G)$ are $G_{\beta\omega}$−open sets in X. Hence X is a $G_{\beta\omega}^2$−space. □

THEOREM 4.13. Let $f : (X, \tau) \to (Y, \rho)$ be $G_{\beta\omega}$−continuous injection function. If f is an open (or closed) function and Y is a regular space then X is a $G_{\beta\omega}$−regular space.

PROOF. Similar for Theorem (3.14). □

THEOREM 4.14. Let $f : (X, \tau) \to (Y, \rho)$ be an injection $G_{\beta\omega}$−continuous function. If f is closed function and Y is a normal space then X is a $G_{\beta\omega}$−normal space.

PROOF. Suppose F and H are any two closed sets in X such that $F \cap H = \emptyset$. since Since f is injection and closed function then $f(F)$ and $f(H)$ are closed sets in Y and

$$f(H) \cap f(F) = f(H \cap F) = f(\emptyset) = \emptyset.$$

Since Y is a normal space then there are two open sets G and U in Y such that $f(F) \subseteq G$, $f(H) \subseteq U$ and $U \cap G = \emptyset$. Since f is injection then $F \subseteq f^{-1}(G)$, $H \subseteq f^{-1}(U)$ and

$$f^{-1}(G) \cap f^{-1}(U) = f^{-1}(G \cap U) = f^{-1}(\emptyset) = \emptyset.$$

Since f is a $G_{\beta\omega}$−continuous then $f^{-1}(G)$ and $f^{-1}(U)$ are $G_{\beta\omega}$−open in X. Hence X is a $G_{\beta\omega}$−normal space. □

5. $\beta\omega$−CONNECTEDNESS PROPERTY

DEFINITION 5.1. Let (X, τ) be a topological space and A, B be two nonempty subsets of X. The sets A and B are called a $\beta\omega$−separated sets if $Cl_{\beta\omega}(A) \cap B = \emptyset$ and $A \cap Cl_{\beta\omega}(B) = \emptyset$.

REMARK 5.2. Let (X, τ) be a topological space. Then

(1) Any $\beta\omega$−separated sets are disjoint sets, since $A \cap B \subseteq A \cap Cl_{\beta\omega}(B) = \emptyset$.

(2) Any two nonempty $\beta\omega$−closed sets in X are $\beta\omega$−separated if they are disjoint sets.

DEFINITION 5.3. A topological space (X, τ) is called a $\beta\omega$−disconnected space if it is the union of two $\beta\omega$−separated sets. Otherwise A (X, τ) is called a $\beta\omega$−connected space.

EXAMPLE 5.4. Any a countable topological space (X, τ) is a $\beta\omega$−disconnected space if X has more that one point.

The proof of the following theorem is clear since $Cl_{\beta\omega}(A) \subset Cl(A)$.

THEOREM 5.5. Every disconnected space is a $\beta\omega$−disconnected space.

The converse of the above theorem need not be true.

EXAMPLE 5.6. In the topological space (X, T), where $T = \{\emptyset, X\}$ and $X = \{a, b\}$, is $\beta\omega$−disconnected space but it is a connected space.

THEOREM 5.7. A topological space (X, τ) is a $\beta\omega$−disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega$−open sets.

PROOF. Suppose that (X, τ) is a $\beta\omega$−disconnected space. Then X is the union of two $\beta\omega$−separated sets, that is, there are two nonempty subsets A and B of X such that

$$Cl_{\beta\omega}(A) \cap B = \emptyset, \ A \cap Cl_{\beta\omega}(B) = \emptyset \text{ and } A \cup B = X.$$

Take $G = X − Cl_{\beta\omega}(A)$ and $H = X − Cl_{\beta\omega}(B)$. Then G and H are $\beta\omega$−open sets. Since $B \neq \emptyset$ and $Cl_{\beta\omega}(A) \cap B = \emptyset$, then $B \subseteq$

,

$X - Cl_{\beta\omega}(A)$, that is, $G = X - Cl_{\beta\omega}(A) \neq \emptyset$. Similar $H \neq \emptyset$. Since $Cl_{\beta\omega}(A) \cap B = \emptyset$, $A \cap Cl_{\beta\omega}(B) = \emptyset$ and $A \cup B = X$, then

$$X - (G \cap H) = (X - G) \cup (X - H) = [Cl_{\beta\omega}(A)] \cup [Cl_{\beta\omega}(B)] = X.$$

That is, $G \cap H = \emptyset$.

Conversely, suppose that (X, τ) is the union of two disjoint nonempty $\beta\omega-$open subsets, say G and H. Take $A = X - G$ and $B = X - H$. Then A and B are $\beta\omega-$closed sets, that is, $Cl_{\beta\omega}(A) = A$ and $Cl_{\beta\omega}(B) = B$. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H) = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

That is, (X, τ) is a $\beta\omega-$disconnected space. $\square$

COROLLARY 5.8. A topological space (X, τ) is a $\beta\omega-$disconnected space if and only if it is the union of two disjoint nonempty $\beta\omega-$closed subsets.

PROOF. Suppose that (X, τ) is a $\beta\omega-$disconnected space. Then by Theorem (5.7), (X, τ) is the union of two disjoint nonempty $\beta\omega-$open subsets, say G and H. Then $X - G$ and $X - H$ are $\beta\omega-$closed subsets. Since $G \neq \emptyset$, $H \neq \emptyset$ and $X = G \cup H$ then $X - G \neq \emptyset$, $X - H \neq \emptyset$ and

$$(X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Since $G \cap H = \emptyset$ then

$$(X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega-$closed subsets. Conversely, suppose that (X, τ) is the union of two disjoint nonempty $\beta\omega-$closed subsets, say G and H. Take $A = X - G$ and $B = X - H$. Then A and B are $\beta\omega-$open sets. Since $H \neq \emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, $A \neq \emptyset$. Similar $B \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_{\beta\omega}(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H) = \emptyset.$$

Similar, $A \cap Cl_{\beta\omega}(B) = \emptyset$. Note that

$$A \cup B = (X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Then by Theorem (5.7), (X, τ) is a $\beta\omega-$disconnected space. $\square$

THEOREM 5.9. A topological space (X, τ) is a $\beta\omega-$connected space if there is no nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed.

PROOF. Suppose that (X, τ) is a $\beta\omega-$connected space. Let A be a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. Then $X - A$ is a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. Since $A \cup (X - A) = X$, then by Theorem (5.7), X is a $\beta\omega-$disconnected space and this a contradiction. So there is no nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed set. Conversely, suppose that (X, τ) is a $\beta\omega-$disconnected space. Then by Theorem (5.7), X is the union of two disjoint nonempty $\beta\omega-$open subsets, say A and B. Then $X - B = A$ is $\beta\omega-$closed subset of X. Since $B \neq \emptyset$ and $X = A \cup B$ then A is a nonempty proper subset of X which is both $\beta\omega-$open and $\beta\omega-$closed. This is a contradiction with the hypothesis. Hence (X, τ) is a $\beta\omega-$connected space. $\square$

THEOREM 5.10. Let $f : (X, \tau) \to (Y, \rho)$ be a $\beta\omega-$continuous surjection function. If X is a $\beta\omega-$connected space then Y is connected space.

PROOF. Suppose that Y is a disconnected space. Then by Theorem (5.7), Y is the union of two disjoint nonempty open subsets, say G and H. Since f is a $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega-$open sets in X. Since $G \neq \emptyset$, $H \neq \emptyset$ and f is a surjection then $f^{-1}(H) \neq \emptyset$ and $f^{-1}(G) \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$ then

$$f^{-1}(G) \cap f^{-1}(H) = f^{-1}(G \cap H) = f^{-1}(\emptyset) = \emptyset$$

and

$$f^{-1}(G) \cup f^{-1}(H) = f^{-1}(G \cup H) = f^{-1}(Y) = X.$$

Hence X is the union of two disjoint nonempty $\beta\omega-$open subsets, that is, X is a $\beta\omega-$disconnected space. This is a a contradiction. Hence Y is a connected space. $\square$

THEOREM 5.11. Let $f : (X, \tau) \to (Y, \rho)$ be a slightly $\beta\omega-$continuous surjection function. If X is a $\beta\omega-$connected space then Y is connected space.

PROOF. Suppose that Y is a disconnected space. Then by Theorem (2.1), Y is the union of two disjoint nonempty open subsets, say G and B. Then G and B are clopen sets in Y. Since f is a slightly $\beta\omega-$continuous then $f^{-1}(G)$ and $f^{-1}(H)$ are $\beta\omega-$open sets in X. Since $G \neq \emptyset$, $H \neq \emptyset$ and f is a surjection then $f^{-1}(H) \neq \emptyset$ and $f^{-1}(G) \neq \emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$ then $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Hence X is the union of two disjoint nonempty $\beta\omega-$open subsets, that is, X is a $\beta\omega-$disconnected space. This is a a contradiction. Hence Y is a connected space. $\square$

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